Math 215 - Problem Set 2: Vector Functions and Space Curves, Derivatives and Integrals of Vector Functions, Arc Length, Motion in Space.

Math 215 SI

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1 Review

1.1 Vector Functions and Space Curves

1.1.1 Vector Function Definition

A vector function is a function that takes a real number as input and outputs a vector. It can be written as $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f(t)$, $g(t)$, and $h(t)$ are scalar functions of t.

1.1.2 Space Curve Definintion

A space curve is the set of all points $\mathbf{r}(t)$ in space as t varies over an interval. It can be thought of as the path traced out by a particle moving in space.

1.1.3 Integrals, Derivatives, and Limits

Consider a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

- The derivative of $\mathbf{r}(t)$ is $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$.
- The integral of $\mathbf{r}(t)$ is $\int \mathbf{r}(t)dt = \langle \int f(t)dt, \int g(t)dt, \int h(t)dt \rangle$.
- The limit of $\mathbf{r}(t)$ as t approaches t_0 is $\lim_{t\to t_0} \mathbf{r}(t) = \langle \lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t) \rangle$.

1.1.4 Tangent Lines

The tangent line to the curve $\mathbf{r}(t)$ at the point $\mathbf{r}(t_0)$ is the line that passes through $\mathbf{r}(t_0)$ and has the same direction as the velocity vector $\mathbf{r}'(t_0)$.

1.2 Arc Length

Given a curve parameterized by $\mathbf{r}(t)$ the arc length L between $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is given by:

$$L = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt$$

1.3 Motion in Space

Given a position vector $\mathbf{r}(t)$ that describes the position of a particle at time t, the velocity vector is $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

2 Problems

Problem 1

(a) Find the arc length of the curve $\mathbf{r}(t) = \langle cos(t), sin(t), \frac{2}{3}t^{\frac{3}{2}} \rangle$ from $t = 0$ to $t = \pi$.

(b) Find the equation of the tangent line to the curve given in part (a) at $t = 0$.

(c) If t is the time, find the speed of the particle at time $t = 2\pi$.

Problem 2

In this problem all coordinates are measured in meters and time is measured in seconds. At time $t = 0$ a ladybug, named Sam, is at position $(1, 1, 1)$ and is flying with constant velocity $< 1, 2, 3 >$ meters per second. A sensor placed at $(3, 6, 7)$ can detect ladybug motion that occurs within a sphere of radius 7 meters. Does the sensor detect Sam? If so, at what time is Sam last detected by the sensor?

Problem 3

The trajectory of a particle is given by $r(t) =< \sqrt{3}t^2, 2t^3, \sqrt{6}t^2 >$. Let C denote the corresponding space curve.

(a) Find an equation of the tangent line to C at the point $(4\sqrt{3}, 16, 4\sqrt{6})$.

(b) How long is C on the interval $0 \leq t \leq \sqrt{8}$?