Math 215 - Problem Set 3: Functions of several variables, Partial Derivatives, Tangent Planes and Linear Approximations

Math 215 SI

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1 Review

1.1 Functions of Several Variables

1.1.1 Function of Serveral Variables Definition

A function of several variables is a function that takes two or more variables as input and produces a single output. For example, a function f of two variables x and y can be written as $f(x, y)$. The domain of f is the set of all pairs (x, y) for which $f(x, y)$ is defined, and the range of f is the set of all possible values of $f(x, y)$.

1.1.2 Level Curves

A level curve is given by $k = f(x, y)$, where k is a constant. It represents the set of all points (x, y) in the domain of f where the function $f(x, y)$ takes on the same value k. Level curves are useful for visualizing functions of two variables, as they provide a way to see how the function behaves in different regions of its domain.

1.1.3 Contour Map

A contour map is a graphical representation of a function of two variables, $f(x, y)$, where contour lines are drawn to connect points that have the same function value. Each contour line represents a specific value of the function, and the spacing between the lines indicates the rate of change of the function. Contour maps are useful for visualizing the topography of a surface, as they provide a way to see how the function values change over the domain.

1.1.4 Contour Surfaces (Extending level curves to higher dimensions)

A contour surface is the three-dimensional analog of a contour line (or level curve). It is a surface in three-dimensional space representing points where a function of three variables $f(x, y, z)$ is constant. For example, the equation $f(x, y, z) = k$ defines a contour surface for a constant k. Contour surfaces are useful for visualizing functions of three variables, as they provide a way to see how the function behaves in different regions of its domain.

1.2 Partial Derivatives

1.2.1 Definition

Partial derivatives are the derivatives of functions of multiple variables with respect to one variable, while keeping the other variables constant. For a function $f(x, y)$, the partial derivative with respect to x is denoted by $\frac{\partial f}{\partial x}$ and is defined as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Similarly, the partial derivative with respect to y is denoted by $\frac{\partial f}{\partial y}$ and is defined as:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial derivatives are used to analyze the rate of change of a function with respect to each of its variables independently.

1.2.2 Theorem

If f_{xy} and f_{yx} are continuous then we have $f_{xy} = f_{yx}$

2 Tangent Planes and Linear Approximations

2.1 Tangent Planes

Given a differentiable function $f(x, y)$, the equation of the tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

where f_x and f_y are the partial derivatives of f with respect to x and y, respectively, evaluated at (x_0, y_0).

2.2 Linear Approximations

The linear approximation (or tangent plane approximation) of a function $f(x, y)$ near a point (x_0, y_0) is given by:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This approximation is useful for estimating the value of the function near the point (x_0, y_0) using the values of the function and its partial derivatives at that point.

3 Problems

Problem 1

Find or estimate, depending on the data provided, the partial derivative in the x direction at the point $(0, 0)$ and the y direction at the point $(0, 0)$ for each of the following functions:

(a) For a function given by the formula $f(x, y) = y^2 \cos(1 + x - y^2 x)$

(b) For a function g described by the data table below

x/y	-2	-1	0	1	2
-2	6	9	9	9	10
-1	12	16	18	19	20
0	20	22	25	27	30
1	28	36	43	47	48
2	35	49	55	61	66

Problem 2

Suppose $g(x, y) = x + ln(5x^2 - 4y^2)$. Find an equation for the tangent plane to the surface given by the equation $z = g(x, y)$ at the point $(1, 1, 1)$.

Problem 3

Suppose that $f(x, y)$ is a differentiable function with continuous derivatives with:

$$f(2, 5) = 7$$
$$f_x(2, 5) = 3$$
$$f_y(2, 5) = -2$$

Consider the curve C given by the intersection of the plane $x = 2$ and the surface $z = f(x, y)$. Find a parametric equation of the line that lies on the plane $x = 2$ and is a tangent line to C at point $(2, 5, 7)$.

Problem 4

Which of the following equations does the function $z = f(x+t) + g(x-t)$ satisfy for all differentiable functions $f(x)$ and $g(s)$ in a single variable?

(a) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 0$

(b) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}$

(c) $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial t^2} = 0$

(d) $\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial t^2}$

(e) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

(f) $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$