Math 215 - Problem Set 4: Maximum and Minimum Values, Local and Global Extrema, and Lagrange Multipliers

Math 215 SI

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1 Review

1.1 Maximum and Minimum Values: Local Extrema

1.1.1 Second Derivative Test

$$H = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

(1) If $H > 0$ and $f_{xx} > 0$, then f has a local minimum at (a, b).

(2) If $H > 0$ and $f_{xx} < 0$, then f has a local maximum at (a, b).

(3) If $H < 0$, then f has a saddle point at (a, b).

(4) If $H = 0$, then the test is inconclusive.

1.2 Maximum and Minimum Values: Global Extrema

1.2.1 Extreme Value Theorem

If f is continuous on a closed and bounded set D, then f has both a maximum and minimum value on D.

(1) Evaluate f at all critical points in D.

(2) Find the maximum and minimum values of f on the boundary of D.

(3) Compare the values from steps (1) and (2) to find the global maximum and minimum values of f on D.

(4) Largest value is the global maximum, smallest value is the global minimum.

1.3 Lagrange Multipliers

To find the maximum and minimum values of a function $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, we solve the system of equations:

(1) $\nabla f = \lambda \nabla g$

(2) $g(x, y, z) = k$

(3) where $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$ and $\nabla g = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$

(4) and λ is the Lagrange multiplier.

(5) The solutions to the system of equations are the critical points of f subject to the constraint $g(x, y, z) = k$.

2 Problems

Problem 1

Find and classify the critical points for the function $h(x, y) = x^4 + y^3 - 6y - 2x^2$.

Problem 2

Find the maximum and minimum values of the function $f(x, y) = x + y$ on the curve $x^2 + y^2 - xy = 4$.

Problem 3

Consider a hypothetical planet whose surface can be approximated by the sphere $x^2 + y^2 + z^2 = 1$. Suppose the planet's temperature is stange and given by $T(x, y, z) = 2x + 2y + z$. Find the hottest and coldest points on the planet.

Problem 4

You are walking on the graph $f(x, y) = 2y\cos(\pi x) + 5$ and you are standing at the point $(1, 1, 4)$

(a) In which direction should you walk in order to stay at a height of 4. Report your answer as a unit vector.

(b) In which direction should you walk in order to decrease your height the fastest? Report your answer as a unit vector.