Math 215 - Problem Set 5: Double Integrals, Rectangles, General Regions, and Polar Coordinates

Math 215 SI

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1 Review

1.1 Double Integrals over Rectangles

For a function $f(x, y)$ defined on a rectangle $R = [a, b] \times [c, d]$, the double integral of f over R is defined as

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij}) \Delta x \Delta y$$

1.1.1 Fubini's Theorem

If $f(x, y)$ is continuous on a rectangle $R = [a, b] \times [c, d]$, then

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

1.2 Double Integrals over General Regions

Let $f(x, y)$ be a continuous function defined on a region D in the xy-plane.

1.2.1 Type I Regions

A region D is a **Type I** region if it is bounded by the graphs of two functions $y = g_1(x)$ and $y = g_2(x)$, and the lines $x = a$ and $x = b$. The double integral of f over D is

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

1.2.2 Type II Regions

A region D is a **Type II** region if it is bounded by the graphs of two functions $x = h_1(y)$ and $x = h_2(y)$, and the lines $y = c$ and $y = d$. The double integral of f over D is

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

1.2.3 Properties

- The area of D is given by

$$\text{Area}(D) = \iint_D 1\, dA$$

- The average value of f over D is given by

$$\text{Avg}(f) = \frac{1}{\text{Area}(D)} \iint_D f(x, y)\, dA$$

- The net volume of the solid bounded by the surface $z = f(x, y)$ and the region D is given by

$$\text{Volume} = \iint_D f(x, y)\, dA$$

- Pay attention to symmetry when setting up double integrals over general regions. If D is symmetric with respect to the x-axis, y-axis, or origin, you can take advantage of this symmetry to simplify the integral.

1.3 Polar Coordinates

1.3.1 Converting Between Rectangular and Polar Coordinates

The conversion formulas between rectangular and polar coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \end{aligned}$$

1.3.2 Double Integrals in Polar Coordinates

If $f(x, y)$ is a continuous function defined on a region D in the xy-plane, then the double integral of f over D can be expressed in polar coordinates as

$$\iint_D f(x,y)\,dA = \iint_E f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where D is the region in the xy-plane that corresponds to the region E in the $r\theta$-plane, and $dA = r\,dr\,d\theta$.

2 Problems

2.1 Problem 1

Consider the double integral

$$\iint xy\, dA$$

over the triangular region D bounded by the three straight lines $y = 0$, $x = 0$, and $x + y = 1$.

(a) What are the three verticies of D.

(b) Evaluate the double integral.

2.2 Problem 2

Compute the following double integral

$$\int_0^{\frac{1}{2}} \int_{y\sqrt{3}}^{\sqrt{1-y^2}} 30xy^2 \, dx \, dy$$

2.3 Problem 3

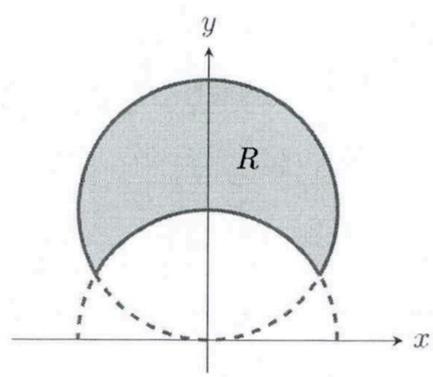
Consider the function $f(x,y) = x - y$ on the region D in $\mathbb{R}^2$ defined by $x + y \geq 0$ and $x^2 + y^2 \leq 4$.

(a) Sketch the region D.

(b) Evaluate the double integral $\iint_D f(x,y)\,dA$.

2.4 Problem 4

Let R be the region inside the circle $x^2 + (y-1)^2 = 1$ and outside the circle $x^2 + y^2 = 1$ pictured below.



(a) Find the bounds of the region R in polar coordinates.

(b) Suppose we have a lamina in the shape of R with a mass density function.

$$\delta(x, y) = \frac{1}{\sqrt{x^2 + y^2}} - \frac{x}{x^2 + y^2}$$

Find the total mass, M, of the lamina.