Math 215 - Problem Set 6: Applications of Double Integrals, Triple Integrals, Triple integrals in Cylindrical Coordinates

Math 215 SI

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1 Review

1.1 Applications of Double Integrals

1.1.1 Mass

Consider a lamina which occupies a region D on the xy-plane. The density of the lamina at a point (x, y) is given by $\delta(x, y)$. The mass of the lamina is given by the double integral

$$\int \int_D \delta(x, y) dA$$

where $dA = dxdy$.

1.1.2 Center of Mass

The center of mass of the lamina is given by the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{M} \int \int_D x\delta(x, y) dA$$

$$\bar{y} = \frac{1}{M} \int \int_D y\delta(x, y) dA$$

and M is the mass of the lamina.

1.1.3 Surface Area

The surface area of a surface S on domain D is given by the double integral

$$\int \int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

where $z = f(x, y)$ is the equation of the surface.

1.2 Triple Integrals

Let f(x,y,z) be a function defined on a volume V in $\mathbb{R}^3$. The triple integral of f over V is given by

$$\int\int\int_V f(x,y,z)dV$$

where $dV = dxdydz$. If V is defined by $a \le x \le b$, $c \le y \le d$, and $e \le z \le f$, then the triple integral can be written as

$$\int_c^d \int_a^b \int_e^f f(x,y,z)dzdxdy$$

1.2.1 Applications

- **Volume:** The volume of a solid occupying a region V in $\mathbb{R}^3$ is given by

$$\int\int\int_V dV$$

- **Mass:** The mass of a solid occupying a region V in $\mathbb{R}^3$ with density $\delta(x,y,z)$ is given by

$$\int\int\int_V \delta(x,y,z)dV$$

- **Center of Mass:** The center of mass of a solid occupying a region V in $\mathbb{R}^3$ is given by the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{1}{M}\int\int\int_V x\delta(x,y,z)dV$$

$$\bar{y} = \frac{1}{M}\int\int\int_V y\delta(x,y,z)dV$$

$$\bar{z} = \frac{1}{M}\int\int\int_V z\delta(x,y,z)dV$$

  and M is the mass of the solid.

1.3 Triple Integrals in Cylindrical Coordinates

In cylindrical coordinates, the triple integral of a function $f(x,y,z)$ over a volume V is given by

$$\int\int\int_V f(x,y,z)dV = \int\int\int_V f(r\cos(\theta), r\sin(\theta), z)rdzdrd\theta$$

where $dV = rdzdrd\theta$.

2 Problems

2.1 Problem 1

Evaluate the triple integral for the following choices of B;

$$\int\int\int_B xyz dV$$

(i) B is the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

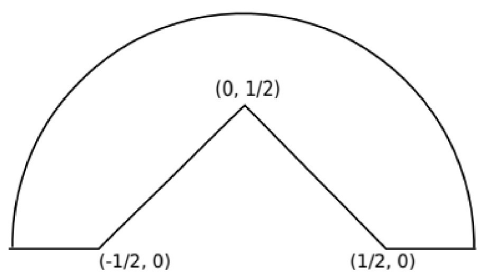
(ii) B is the region $0 \leq x \leq y \leq z \leq 1$.

2.2 Problem 2

(i) Consider the half-disc $0 \leq x^2 + y^2 \leq 1$, $y \geq 0$. Assume that the density is $\rho(x, y) = 1$. Find $\bar{y}$, the y-coordinate of the center of mass of the half-disc.

(ii) Find the y-coordinate of the center of mass of the triangular region with vertices pictured below and assume the same density from part (i).

(iii) Suppose the triangular wedge is removed from the half-disc to get the region pictured below. Find the new y-coordinate of the center of mass of the new region.



2.3 Problem 3

Find the volume of the solid bounded by the following two paraboloids using cylindrical coordinates.

$$z = x^2 + y^2$$

$$z = 2 - x^2 - y^2$$