Math 215 - Problem Set 6: Vector Fields, Line Integrals, FT for Line Integrals

Math 215 SI

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1 Review

1.1 Vector Fields

A vector field is a function that assigns a vector to each point in space. A vector field $\mathbf{F}$ in $\mathbb{R}^3$ can be represented as:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

where P, Q, and R are scalar functions of x, y, and z.

(i) $\mathbf{F}$ is continuous if and only if coordinate functions P, Q, and R are continuous.

(ii) $\mathbf{F}$ is differentiable if and only if coordinate functions P, Q, and R are differentiable.

1.2 Line Integrals

Let C be a smooth curve parameterized by $(r(t))$ on $[a, b]$.

(i) The line integral of a scalar function f along the curve C is given by:

$$\int_C f(x, y, z)ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$$

where $ds = \|\mathbf{r}'(t)\|dt$ is the differential arc length.

(ii) The line integral of a vector field $\mathbf{F}$ along the curve C is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

where $d\mathbf{r} = \mathbf{r}'(t)dt$ is the differential vector along the curve. Note that if $\mathbf{F}$ is a force field, then the line integral represents the work done by the force field along the curve C.

1.3 Fundamental Theorem for Line Integrals

Let C be a smooth curve parameterized by $(r(t))$ on $[a, b]$. If $\mathbf{F}$ is a conservative vector field, then there exists a scalar potential function f such that $\nabla f = \mathbf{F}$. The fundamental theorem for line integrals states that:

(i) The line integral of $\mathbf{F}$ along the curve C is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where f is the potential function for $\mathbf{F}$.

(ii) The line integral is independent of the path taken from $\mathbf{r}(a)$ to $\mathbf{r}(b)$. And if C is a closed curve, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

1.3.1 When is F conservative?

A vector field $\mathbf{F}$ is conservative if and only if the following conditions are satisfied:

(i) The vector field is defined on a simply connected domain.

(ii) The curl of the vector field is zero:

$$\nabla \times \mathbf{F} = 0$$

In two dimensions this is to say that

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x}$$

2 Problems

2.1 Problem 1

Consider the force field $\mathbf{F}(x, y) = (2x - y, 4y - x)$

(i) Show that $\mathbf{F}$ is conservative.

(ii) Find a potential function f such that $\nabla f = \mathbf{F}$.

(iii) Find the work done by $\mathbf{F}$ along the path C_1 from $(1, 0)$ to $(2, 1)$

(iv) Find a curve C_2 such that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2$.

2.2 Problem 2

Consider the vortex field $\mathbf{V}(x, y) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$

(i) Is $\mathbf{V}$ conservative on the domain $y > 0$?

(i) Find $\int_C V \cdot dr$ where C is given by $x^2 + (y-2)^2 = 1$

2.3 Problem 3

Find the work done by the force field $F(x, y, z) = (x + y, y^2 - z, 2z)$ along the line segment C from $(0, 0, 1)$ to $(2, 1, 0)$.